Quotients of a Universal Locally Projective Polytope of type $\{5, 3, 5\}$ *

Michael I. Hartley
KDU College
Jalan SS 22/41, Damansara Jaya
Selangor, 47400, Malaysia
michael@kdu.edu.my

Dimitri Leemans
Département de Mathématiques
Université Libre de Bruxelles
C.P.216 – Géométrie
Boulevard du Triomphe
B-1050 Bruxelles
dleemans@ulb.ac.be

In Memory of H. S. M. "Donald" Coxeter, 1907–2003

Abstract

This article examines the universal polytope $\mathcal{P}$ (of type $\{5, 3, 5\}$) whose facets are dodecahedra, and whose vertex figures are hemi-icosahedra. The polytope is proven to be finite, and the structure of its group is identified. This information is used to classifiy the quotients of the polytope. A total of 145 quotients are found, including 69 section regular polytopes with the same facets and vertex figures as $\mathcal{P}$.

1 Introduction

This article may be seen in three different ways. On the one hand, it was inspired by, and solves, a problem in the theory of polytopes. A certain

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regular polytope is proven to exist and be finite, and its quotient polytopes are discovered. Being about polytopes, it is also a paper about geometries or buildings, since (for example) every regular polytope is a thin regular geometry with a string diagram. Finally, it may also be regarded as an article about Coxeter groups. A certain quotient of a hyperbolic Coxeter group is proven to be finite, and its structure as a group is identified.

The polytope studied is the universal polytope $\mathcal{P}$ whose facets are dodecahedra and whose vertex figures are hemi-icosahedra. This polytope is locally projective, that is, it is section regular with facets and vertex figures either spherical or projective, and not both spherical. The polytope, of type $\{5, 3, 5\}$, covers every polytope with facets and vertex figures of these types. A classification of its quotients is therefore important since it encompasses a classification of the locally projective polytopes of type $\{5, 3, 5\}$ – every such polytope or its dual is a quotient of $\mathcal{P}$. The automorphism group of this polytope is the quotient of the Coxeter group $[5, 3, 5] = \langle s_0, s_1, s_2, s_3\rangle$ by the normal subgroup generated by all quotients of $(s_1s_2s_3)^5$.

The results of the article is expressed mainly in the language of abstract polytopes, since that is the setting which gave rise to the results, and (in the authors' opinion) in which they seem the most significant. It is assumed from this point on that the reader is familiar with the basic concepts of abstract polytopes and their quotients. The most important reference in the theory of abstract regular polytopes is [13]. An abstract polytope is a poset satisfying certain properties that are satisfied by the face-lattices of classical polytopes, as well as honeycombs in euclidian and hyperbolic space, and other objects. The theory therefore encompasses all of the latter, as well as a rich assortment of polytopes on other spaceforms and objects for which it is difficult to define a natural topology.

The locally projective polytopes fall into the latter class. A (section) regular polytope may be defined to be *locally* X if its minimal nonspherical sections have topological type X (see [11]). Alternatively but not equivalently, some authors define a locally X polytope to be one whose facets and vertex figures are either spherical or X, but not both spherical (see for example [15]). Note that the former definition subsumes the latter, and that in rank 4 the two definitions are equivalent, since all polytopes of rank 2 or less are spherical.

A regular polytope is one whose automorphism group acts transitively on its set of flags (that is, maximal totally ordered subsets). The most important result in the study of abstract regular polytopes is that they are in one to one correspondence with so-called string C-groups, that is groups generated by involutions $s_0, s_1, \ldots, s_{n-1}$ where first of all $s_is_j = s_js_i$ whenever $i \neq j, j\pm1$, and secondly, $\langle s_i : i \in I\rangle \cap \langle s_i : i \in J\rangle = \langle s_i : i \in I \cap J\rangle$ for any $I, J \subseteq \{0, \ldots, n-1\}$. The automorphism groups of regular polytopes

are always C-groups, and from any C-group the corresponding polytope may be reconstructed.

Given a polytope $\mathcal{P}$ and a subgroup N of its automorphism group, the elements of $\mathcal{P}$ are partitioned into orbits $\{F \cdot N : F \in \mathcal{P}\}$ by N. We may define a poset $\mathcal{P}/N$ on these orbits, the *quotient* of $\mathcal{P}$ by N, in a natural way, letting $F \cdot N \leq G \cdot N$ if $F \leq G$ (that is, $F \cdot N \leq G \cdot N$ if and only if there exists $F' \in F \cdot N$ and $G' \in G \cdot N$ with $F' \leq G'$). When $\mathcal{P}$ is regular, the conditions on N for which $\mathcal{P}/N$ is again a polytope are well-known (see [12], or Section 2D of [13]).

In fact, every polytope may be written as a quotient of some regular polytope (see [4] and [5]), and the quotients of a regular polytope $\mathcal{P}$ are in one-to-one correspondence with conjugacy classes of so-called "semisparse" subgroups of $\mathrm{Aut}(\mathcal{P})$.

Let $\mathcal{K}$ and $\mathcal{L}$ be regular polytopes. If there exists a polytope $\mathcal{Q}$ whose facets are of type $\mathcal{L}$ and whose vertex figures are of type $\mathcal{K}$, then there is a *universal* such polytope $\mathcal{P}$, denoted $\{\mathcal{L}, \mathcal{K}\}$ (with automorphism group $[\mathcal{L}, \mathcal{K}]$), which "covers" all other such polytopes in the sense that they are quotients of $\mathcal{P}$. This was shown in Theorem 2.5 of [7], and much earlier for the case when $\mathcal{Q}$ is regular in [14] (see Theorem 4A2 of [13]). The search for polytopes with particular facets and vertex figures therefore usually follows the following pattern. First the universal such polytope is discovered (if not already known). Secondly, the quotients of this universal polytope are sought. In [7], various results about semisparse subgroups were uncovered that facilitate this process.

Let $W = \langle s_0, \ldots, s_{n-1} \rangle$ be the group of a regular polytope $\mathcal{P}$, let $H_{n-1} = \langle s_0, \ldots, s_{n-2} \rangle$ be the group of its facets, and let $H_0 = \langle s_1, \ldots, s_{n-1} \rangle$ be the group of its vertex figures. The key result from [7] that we shall use here is as follows. If the vertex figures of $\mathcal{P}$ have no proper quotients, then subgroups N of W are semisparse if and only $N^w \cap H_0 H_{n-1}$ is semisparse in H_{n-1} for all conjugates N^w of N in W (see Theorem 2.7 of [7]).

In this article, we are interested in polytopes whose facets are dodecahedra, and whose vertex figures are hemi-icosahedra. If any such polytopes exist at all, they are quotients of a universal such polytope with automorphism group $W = \langle s_0, s_1, s_2, s_3 \rangle$, satisfying $s_0^2 = s_1^2 = s_2^2 = s_3^2 = 1$, $(s_0 s_1)^5 = (s_1 s_2)^3 = (s_2 s_3)^5 = 1$, $s_0 s_2 = s_2 s_0$, $s_1 s_3 = s_3 s_1$, $s_0 s_3 = s_3 s_0$ and $(s_1 s_2 s_3)^5 = 1$. For the remainder of this article, W shall denote this group. It is shown in Section 2 that W is a C-group, the group of a well-defined polytope $\mathcal{P}$ whose facets and vertex figures are as desired. This is done by exhibiting an example of another (smaller) such polytope, which must therefore be a proper quotient of $\mathcal{P}$.

In Section 3, it is shown that W (and therefore $\mathcal{P}$) is in fact finite, and is the direct product of two large simple groups, the Janko group J_1 and the

projective special linear group $L_2(19)$. In Section 4, the remaining quotients are discovered and tabulated.

Let $H_0 = \langle s_1, s_2, s_3 \rangle$ and $H_3 = \langle s_0, s_1, s_2 \rangle$.

2 Two Quotients of $\mathcal{P}$.

Coxeter, in [2], discovered a self-dual locally projective polytope $\mathcal{P}''$ with 57 hemi-dodecahedral facets (it was also constructed in [16]). We shall call this polytope the *57-cell*. In [6] it was shown that Coxeter's 57-cell has no proper quotients. Its group W'' is generated by $\{s_0'', s_1'', s_2'', s_3''\}$ with the relations of $[5, 3, 5]$ and the additional relations $(s_0''s_1''s_2'')^5 = (s_1''s_2''s_3'')^5 = 1$. Moreover W'' is isomorphic to the simple group $L_2(19)(= PSL_2(19))$, of order $3420 = 2^2.3^2.5.19$. It is a quotient of $W = \langle s_0, s_1, s_2, s_3 \rangle$ by the normal subgroup N'' generated by all conjugates of $(s_0s_1s_2)^5$. The simple group $L_2(19)$ has a permutation presentation on 20 points. Readers interested to see this permutation presentation should download this article's "auxiliary information" available at [8]. (See also the notes at the end of Section 4.)

For the remainder of this article, let $\omega = (s_0s_1s_2)^5$, so that $N'' = \langle\langle \omega \rangle\rangle$. The action of ω (as an automorphism) on $\mathcal{P}$ is to move a (base) flag Φ to the "opposite" flag of the facet contained in Φ.

Define $\nu = \omega s_3$, and let $N' = \langle\langle \nu^3 \rangle\rangle$ be the normal subgroup of W generated by all conjugates of ν^3. The group $W' \cong W/N'$ may be taken to be the group $\langle s_0', s_1', s_2', s_3' \rangle$ whose generators satisfy all the relations of $[5, 3, 5]$ as well as the additional relations $(s_1's_2's_3')^5 = ((s_0's_1's_2')^5 s_3')^3 = 1$. A computer algebra package [3] was used to analyse the group W'. It was found to be isomorphic to the Janko group J_1, a sporadic finite simple group of order $175560 = 2^3.3.5.7.11.19$. It was also checked that W' is a C-group, that $H_3' = \langle s_0', s_1', s_2' \rangle$ has order 120, that $H_0' = \langle s_1', s_2', s_3' \rangle$ has order 60, and therefore that W' is the group of a regular polytope $\mathcal{P}'$ with dodecahedral facets and hemi-icosahedral vertex figures.

The simplest permutation representation for J_1 is a permutation action on 266 points (see [9] and [10] for more information). Again, readers interested in an example of this permutation action are referred to the auxiliary information for this article. (See the notes at the end of Section 4 for details.)

This polytope $\mathcal{P}'$ has 1463 dodecahedral facets, and twice that number of vertices. It is instructive to classify the quotients of $\mathcal{P}'$. Theorem 2.7 of [7] applies, so a subgroup $K \leq W'$ is semisparse if and only if $K^w \cap H_0'H_3' = \{1\}$ or $\{1, (s_0's_1's_2')^5\}$. In J_1, all elements of order 2 are conjugate. It follows that if K contains $(s_0's_1's_2')^5$, it has a conjugate containing s_0', which would contradict the semisparseness of K. Therefore, $\mathcal{P}'/K$ cannot have any hemi-

dodecahedral facets – all its facets must be dodecahedra, and $K^w \cap H_0' H_3' = \{1\}$ for all conjugates K^w of K in W'.

The Sylow p-subgroups of any group are all conjugate. For $p = 3$, 5, 7, 11 or 19, the Sylow p-subgroups of J_1 are cyclic of order p. If $H_0' H_3'$ contains an element of order p, then K can not, otherwise there would exist a conjugate of K intersecting $H_0' H_3'$ nontrivially. It may be verified, however, that $H_0' H_3'$ does contain elements of each of these orders: for example $s_1' s_2'$ has order 3, $s_0' s_1'$ has order 5, $s_3' s_1' s_2' s_1' s_0'$ has order 19, $s_3' s_2' s_1' s_0' s_1' s_0'$ order 11 and $s_1' s_3' s_2' s_1' s_3' s_0' s_1' s_0' s_2' s_1' s_0'$ order 7. Therefore the polytope $\mathcal{P}'$, like $\mathcal{P}''$, has no proper quotients.

3 The Group Structure of W.

We can now investigate the structure of the group $W = \langle s_0, s_1, s_2, s_3 \rangle$. An important result about W is the following.

3.1 THEOREM *W is a C-group, and the polytope $\mathcal{P}$ has dodecahedral facets and hemi-icosahedral vertex figures.*

Proof: Proposition 4A8 of [13], combined with either of the examples of the previous section, show that the universal polytope $\{\{5,3\},\{3,5\}_5\}$ exists, and that the presentation of its automorphism group is just that of W. □

We already know that W has two normal subgroups N' and N'', of index 175560 and 3420 respectively. This information may be used to discover the structure of W.

3.2 THEOREM *$W \cong J_1 \times L_2(19)$.*

Proof: W has a subgroup $L = \langle s_2 s_1, (s_0 s_1)^2 s_2 s_3 s_0, s_0 s_1 s_0 s_3 s_2 s_3 s_1 \rangle$ of index 20. That the index is 20 may be verified using the Todd-Coxeter coset enumeration technique, either by hand or using a computer.

Let x_1, x_2 and x_3 be the three generators of L in the order given, and let $x = x_2^{-1} x_3 x_1$ and $y = x_1 x_3^{-1}$. Then x and y generate L, and satisfy the relations $y^3 = xyxy^{-1}xyx^{-1}yx = 1$. This may be shown either laboriously by hand, or by using a computer. Finally, it may be shown that the group $\langle x, y : y^3 = xyxy^{-1}xyx^{-1}yx = 1 \rangle$ is finite, of order 30020760. Since L is a quotient of this group, it follows that W has order at most $20 \times 30020760 = 3420 \times 175560$. In particular, W is finite. On the other hand, since W has quotients isomorphic to the simple groups $L_2(19)$ and J_1, we deduce that W has order at least 3420×175560, so in fact this number is exactly the order of W, and L equals $\langle x, y : y^3 = xyxy^{-1}xyx^{-1}yx = 1 \rangle$.

The details of this derivation of the structure of W were obtained using GAP version 4 (see [3]). In particular, the neat presentation for L was obtained in GAP using the Reduced Reidmeister-Schreier method, followed by Tietze Transformations to simplify the presentation. This gave $L = \langle x, y : y^3 = xyxy^{-1}xyx^{-1}yx = 1\rangle$ directly.

The composition series of W contains the Janko group J_1 and the projective special linear group $L_2(19)$ – these are the quotients of W by (respectively) the normal subgroups N' and N'' of Section 2. The composition series cannot contain any other factors, since the order of W is exactly the product of the orders of J_1 and $L_2(19)$. It follows that $N' \cong L_2(19)$ and $N'' \cong J_1$.

Since $|W| = |N'|.|N''|$, and since N' and N'' are both normal in W, it follows that if $W = N'N''$ then W is an internal direct product $N' \times N''$. Note in any case that $N'N''$ is closed under the group multiplication. To show $W = N' \times N''$ it therefore suffices to show that the generators of W are found in $N'N''$.

Now N' is generated by all conjugates of ν^3, and N'' by all conjugates of ω. However, $\nu^3 = (\omega s_3)^3 = s_3(s_3\omega s_3)\omega(s_3\omega s_3)$. Therefore, the generator s_3 of W may be expressed as a product $\nu^3(s_3\omega s_3)\omega(s_3\omega s_3)$ of an element of N' with an element of N''. Since $s_2 = (s_3s_2s_3s_2)s_3(s_2s_3s_2s_3)$, $s_1 = (s_2s_1)s_2(s_1s_2)$ and $s_0 = (s_1s_0s_1s_0)s_1(s_0s_1s_0s_1)$, that is, all generators of W are mutually conjugate, it follows that each generator of W may be written $\alpha'\alpha''$ for some conjugate α' of ν^3 and some conjugate α'' of $(s_3\omega s_3)\omega(s_3\omega s_3)$. Therefore $W = N' \times N''$ as required. □

The twenty right cosets of L are L, Ls_0, $Ls_1 = Ls_2$, Ls_3, Ls_0s_1, Ls_0s_2, Ls_0s_3, Ls_1s_3, Ls_3s_2, $Ls_0s_1s_0$, $Ls_0s_1s_2$, $Ls_0s_1s_3$, $Ls_0s_2s_1$, $Ls_0s_2s_3$, $Ls_0s_3s_2$, $Ls_1s_3s_2$, $Ls_3s_2s_1$, $Ls_3s_2s_3$, $Ls_0s_1s_0s_2$ and $Ls_0s_3s_2s_3$. Numbering the cosets in the order given yields a representation of the permutation action of W on these cosets, that is, a homomorphism from W to S_{20}:

$$s_0 \mapsto (1,2)(3,6)(4,7)(5,10)(8,14)(9,15)(11,19)(12,16)(13,17)(18,20),$$

$$s_1 \mapsto (1,3)(2,5)(4,8)(6,13)(7,12)(9,17)(10,15)(11,18)(14,19)(16,20),$$

$$s_2 \mapsto (1,3)(2,6)(4,9)(5,11)(7,15)(8,16)(10,19)(12,14)(13,18)(17,20),$$

$$s_3 \mapsto (1,4)(2,7)(3,8)(5,12)(6,14)(9,18)(10,16)(11,17)(13,19)(15,20).$$

This is not a faithful action of W of course. In fact, these permutations generate a group isomorphic to $L_2(19) \cong W/N'' \cong N'$.

Since $W \cong J_1 \times L_2(19)$, it has a permutation presentation on 286 points, which may be derived from the presentations for W' and W'' of the previous section, via $s_i = s_i's_i''$. This presentation facilitates machine computation.

As before, see the notes at the end of Section 4 for infomation on obtaining this presentation.

Note that $\nu^6 = 1$, since $\nu^6 = (\nu^3)^2 = (\omega(s_3\omega s_3))^3$ and is therefore an element of $N' \cap N''$.

4 The Remaining Quotients of $\mathcal{P}$.

By Theorem 2.7 of [7], the semisparse subgroups N of W may be characterised by the property that $N^w \cap H_0H_3 \subseteq \{1, \omega\}$ for all $w \in W$. This is because any subgroup satisfying this property is semisparse, and conversely any semisparse subgroup of W satisfies this property (see Theorem 2.7 of [7]).

The group $J_1 \times L_2(19)$ has 1262 conjugacy classes of subgroups. It is relatively straightforward to check them one by one to see if they satisfy the above property. This was done, using Magma [1]. The program took approximately 40 days of computing time, on two Intel Xeon processors running at 2GHz with 3Gb of RAM. The authors would like to suggest that the results could have been obtained faster if they had troubled to optimise the code better. In particular, in [7] it was noted that if a C-group (such as W) satisfies the conditions of Theorem 2.7 of [7], then whenever N is semisparse in W, all its subgroups are also be semisparse. This property was not used, but could have saved a significant amount of computation time.

A total of 145 conjugacy classes of semisparse subgroups were discovered, yielding 145 polytopes, most of them new. Further analysis of the semisparse subgroups, to identify for example the facet types and automorphism groups of the polytopes, was done using GAP version 4 release 3 ([3]).

As just mentioned, if N is semisparse and $K \leq N$, then K is also semisparse. Furthermore, $\mathcal{P}/K$ is a cover for $\mathcal{P}/N$. For these reasons, it is important to know the subgroup relations between the 145 semisparse subgroups of W. In particular, it is useful to know the "maximal" semisparse subgroups, that is, those semisparse subgroups which are not proper subgroups of any other. These subgroups are important because the semisparse subgroups of W are exactly the subgroups of these maximal ones.

W has 30 maximal semisparse subgroups. These, with their generating sets, are listed in Table 1. In the table, $v_1 = \nu = (s_0s_1s_2)^5s_3$, $v_2 = v_1s_2$, and $v_6 = v_5s_0 = v_4s_1s_0 = v_3s_0s_1s_0 = v_2(s_1s_0)^2$. The notation used in the "Group" column is as follows: $G \times H$ is the direct product of G with H, and $G : H$ is a semidirect product. G^k means the direct product of k copies of G. Also, as usual, C_k (or just 'k') means the cyclic group of order k, D_{2k} the dihedral group of order $2k$, and A_k and S_k the alternating and

symmetric groups on k points. Finally, as before, J_1 is the first Janko group, and $L_2(p)$ is the projective special linear group of rank 2 over $GF(p)$. The numbering of the groups is in accordance with the ordering of the semisparse subgroups as returned by the algorithms used by the authors. This ordering is descending in the size of the group.

Of the 145 semisparse subgroups, 70 yield section regular polytope. Of these, 69 have dodecahedral facets, and one (namely, number 1) is Coxeter's 57-cell, self-dual, with hemidodecahedral facets. A list of these may be found in Tables 2 and 3. In those tables is listed the number of the group, the isomorphism type of the group, its maximal proper semisparse subgroups in W, the number of facets of the polytope, and the isomorphism type of its automorphism group. Note that group number 40 is not isomorphic to either group 39 or 38, nor any other group labelled $19:9$ in this paper. It is perhaps better described as a non-split extension of a normal subgroup C_{57} by a group of order 3.

The remaining 75 groups (or polytopes) are listed in Tables 4 and 5. Those two tables list similar information to that given for the section regular polytopes, the key difference being in the information given about the facets of the polytopes. An entry in the "Facets" columns of the form D^dH^h means that the polytope has, as facets, d dodecahedra and h hemidodecahedra.

It should not seem surprising or contradictory that (for example) factoring out by a single element of order 2 (namely ω) yields a polytope with 3420 hemi-dodecahedral facets (polytope number 143). For example, factoring the cube, with group $\langle\sigma_0, \sigma_1, \sigma_2\rangle$, by the semisparse subgroup $\{1, (\sigma_0\sigma_1)^2\}$ yields a digonal prism, a polytope with not one, but two digons corresponding to "opposite" squares of the original cube. Likewise here, if a "cusp" is introduced into the "space" occupied by $\mathcal{P}$ by factoring out ω, this forces another 3419 cusps to also form. The elements of $N' = \langle\langle\nu^3\rangle\rangle \cong L_2(19)$ permute among the cusps. The automorphism group A_5 of the hemi-icosahedron also acts on this "space" by rotating it around the cusps, just as the group of the digon acts on the digonal prism in a way that maps each digon to itself. This is why the automorphism group of polytope 143 is $A_5 \times L_2(19)$.

As mentioned earlier, some auxiliary information for this article is available. The auxiliary information is in the form of a file (`g.txt`) containing GAP commands that construct the permutation presentations for the groups of the polytopes $\mathcal{P}'$ and $\mathcal{P}''$, and combine them into a presentation (on 286 points) for W.

The file also contains permutation presentations for representatives of all the 1262 conjugacy classes of subgroups of $W \cong J_1 \times L_2(19)$. The file defines a list (`cl`) of indices to identify which of these groups are semisparse in W. Importing this file into GAP (via, for example, a `Read` command)

places in GAP's workspace a list (geo) consisting of one representative of each conjugacy class of semisparse subgroups of W.

The file g.txt is available via a link from [8]. If that web page should move to a different URL, it should in any case be locatable by searching the web for the title of this article.

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No.	Group	Generators
1	J_1	$v_1^2, (v_2v_1)^2, v_3^5, (v_5v_2v_1)^2$
2	$L_2(19)$	$(v_2v_1^2)^2, (v_6^2v_4v_1)^2$
3	$19^2:9$	$v_1^2v_2v_4^2v_6v_3, v_3^2v_1v_4v_5^2v_6v_1$
4	$19^2:9$	$v_5v_1v_5v_4v_1v_2v_4v_1, v_2^2v_1v_5v_3^2v_2v_4$
5	$D_{38}\times 19:3$	$v_1v_3v_4^2v_5v_2^2v_1, v_1v_6^2v_2v_3v_5v_2v_4$
6	$19\times 11:10$	$v_1v_3v_6v_5^2v_3, v_2^2v_6^2v_1v_6v_2$
7	$2^3\times 19:9$	$v_3v_4v_1v_6v_5v_3v_1, v_5v_6v_5v_1v_2v_5^2, v_6v_2^2v_6v_4v_1v_4v_2$
8	$19\times A_5$	$v_5v_6^2v_3v_5v_1, v_3^2v_2v_3v_1v_2v_3, v_3v_5v_2v_6v_5v_3v_6$
9	$19\times A_5$	$v_5v_6^2v_3v_5v_1, v_2v_3^3v_2v_3v_1$
11	$19\times 19:3$	$v_2v_5v_2v_1v_3v_4v_3, v_4^2v_1v_6v_5v_3v_2v_1^2, v_1v_3v_4v_5v_6v_1v_4v_3v_1$
13	$5\times 19:9$	$v_1v_3^6v_1^2, v_5v_3v_5v_6v_4v_6^2v_2v_1$
16	$2^2:(19:9)$	$v_4v_6^2v_2v_3v_4^2v_2, v_1v_5v_6v_5^2v_6v_3v_4$
17	$2^2:(19:9)$	$v_1v_2v_3v_6v_1v_4v_2v_1, v_4v_6^2v_2v_3v_4^2v_2$
19	$19\times D_{30}$	$v_6v_3v_4v_6v_5^2, v_6v_2^3v_6^2v_1^3, v_5v_6^2v_5v_3v_5v_4v_2v_1$
20	$C_{95}\times S_3$	$v_3^2v_5^2v_6v_5v_4v_1, (v_6v_1)^2v_2v_5v_4v_5v_1$
22	$7\times A_5$	$v_1v_5v_4v_2, v_5v_1v_3v_5v_1v_4v_2^2v_6$
23	$7\times A_5$	$v_1^3, v_4v_3v_4v_2v_6v_3v_2^2v_4, v_4v_6v_5v_4^2v_3v_5v_3v_4$
25	$19\times D_{20}$	$v_3v_1^2v_5^2v_6v_4v_5, v_5v_6^2v_3v_5v_4v_2v_5v_4$
29	$(2^3:7)\times 5$	$(v_3v_1)^2v_5v_2v_1v_4, v_6^2v_3v_4v_5^2v_4v_5$
30	$19\times A_4$	$v_3^6v_1^3, v_6v_3^2v_4v_3^2v_1v_4v_1$
33	$11\times D_{20}$	$v_4v_5v_4v_3v_1^2, v_2v_1^3v_2^4$
42	$(2^3:7)\times 3$	$v_1v_5v_3v_1v_4v_6^2v_1^2, v_3v_6v_2v_6v_4v_6^3v_1$
48	$19\times S_3$	$v_6v_4v_3v_4v_1v_5v_2v_1v_6v_1, v_3v_6v_5v_2v_3v_4^2v_5v_6v_1$
54	$5\times D_{22}$	$v_3v_4v_1v_5v_2^2v_4, v_1^2(v_2^2v_1)^2$
58	$5\times D_{14}$	$v_6v_1^2v_2v_5v_6v_3v_4v_1, v_6v_1^2v_3v_5^2v_1^2v_4v_1$
64	$7:9$	$v_5v_3v_2v_5v_6v_1(v_3v_1)^2, v_2^2v_4v_1^2v_4v_3v_6v_3v_2$
70	$3\times D_{20}$	$v_2^3v_6^2v_1v_6, v_5v_3v_5v_4v_2v_3v_4v_1v_4v_2, v_1v_5v_1v_3v_5v_2v_1v_6v_5v_6$
81	$3\times D_{14}$	$v_5v_3v_1v_2v_4v_2v_1v_6v_1, (v_4v_5v_4)^3$
95	$5\times S_3$	$v_1v_2v_1v_4v_5^2v_4v_6v_4, v_5v_3v_5v_4v_5v_6v_4v_2v_6v_1^2$
127	C_{10}	$v_1v_2v_3v_1v_2v_4v_1^2v_3^2v_1$

Table 1: The Maximal Semisparse Subgroups of $W=[\{5,3\},\{3,5\}_5]$.

No.	Group	Subgroups	#Facets	Aut($\mathcal{P}$)
1	J_1	18, 41, 44, 45, 51, 65, 80	57	$L_2(19)$
2	$L_2(19)$	38, 66, 67, 103, 109	1463	J_1
3	$19^2 : 9$	10, 39, 40	1540	C_6
4	$19^2 : 9$	10, 39, 40	1540	C_6
10	$19 \times 19 : 3$	26, 71, 72	4620	$3^2 \times 2$
11	$19 \times 19 : 3$	26, 73, 74	4620	C_{18}
12	$19 \times 11 : 5$	34, 56, 76	4788	C_{18}
13	$5 \times 19 : 9$	38, 56, 78	5852	D_{12}
22	$7 \times A_5$	57, 62, 66, 82	11913	C_6
23	$7 \times A_5$	57, 61, 67, 82	11913	C_6
26	19^2	106, 107, 108	13860	$C_6 \times C_9$
28	C_{285}	56, 74, 112	17556	$2^2 \times C_9$
30	$19 \times A_4$	58, 72, 116	21945	C_6
33	$11 \times D_{20}$	52, 53, 55, 79, 103	22743	C_{10}
34	C_{209}	107, 119	23940	C_{90}
38	$19 : 9$	71, 128	29260	J_1
39	$19 : 9$	71, 129	29260	$3 \times D_{10}$
40	$19 : 9$ ($\cong 57.3$)	72, 129	29260	C_6
48	$19 \times S_3$	72, 86, 134	43890	C_6
52	$11 \times D_{10}$	77, 100, 121	45486	$2^2 \times 5$
53	$11 \times D_{10}$	77, 100, 122	45486	$2^2 \times 5$
55	C_{110}	77, 100, 126	45486	$2^2 \times 5$
56	C_{95}	107, 137	52668	$9 \times D_{12}$
57	$7 \times A_4$	96, 102, 116	59565	C_6
58	$2^2 \times 19$	86, 139	65835	$3^2 \times 2$
61	$7 \times D_{10}$	89, 114, 121	71478	$2^2 \times 3$
62	$7 \times D_{10}$	89, 114, 122	71478	$2^2 \times 3$
64	$7 : 9$	102, 129	79420	C_6
66	A_5	116, 122, 134	83391	J_1
67	A_5	116, 121, 134	83391	J_1
71	$19 : 3$	107, 141	87780	$3 \times J_1$
72	C_{57}	106, 141	87780	$C_6 \times S_3$
73	$19 : 3$	106, 142	87780	$2 \times L_2(19)$
74	C_{57}	107, 142	87780	$9 \times D_{20}$
76	$11 : 5$	119, 137	90972	$2 \times L_2(19)$
77	C_{55}	119, 138	90972	$2^3 \times 5$
78	C_{45}	111, 128	111188	$2 \times D_{12}$
79	$2^2 \times 11$	100, 139	113715	C_{30}
82	$7 \times S_3$	102, 114, 134	119130	C_6
86	C_{38}	106, 144	131670	$C_6 \times D_{10}$

Table 2: Semisparse Subgroups Corresponding to Section Regular Polytopes - Part 1

No.	Group	Subgroups	#Facets	Aut($\mathcal{P}$)
89	C_{35}	131, 138	142956	$2^3 \times 3$
96	$2^2 \times 7$	114, 139	178695	$3^2 \times 2$
100	C_{22}	119, 144	227430	$C_{10} \times D_{10}$
101	$7 : 3$	131, 142	238260	$2 \times L_2(19)$
102	C_{21}	131, 141	238260	$C_6 \times S_3$
103	D_{20}	121, 122, 126, 139	250173	J_1
106	19	145	263340	$C_6 \times L_2(19)$
107	19	145	263340	$C_9 \times J_1$
108	19	145	263340	$19 : 3$
109	$9 : 2$	128, 134	277970	J_1
111	C_{15}	137, 141	333564	$S_3 \times D_{12}$
112	C_{15}	137, 142	333564	$2^2 \times L_2(19)$
114	C_{14}	131, 144	357390	$C_6 \times D_{10}$
116	A_4	139, 141	416955	J_1
119	11	145	454860	$C_{10} \times L_2(19)$
121	D_{10}	138, 144	500346	$2 \times J_1$
122	D_{10}	138, 144	500346	$2 \times J_1$
126	C_{10}	138, 144	500346	$2 \times J_1$
127	C_{10}	137, 144	500346	$D_{10} \times D_{12}$
128	C_9	141	555940	$2 \times J_1$
129	C_9	141	555940	$S_3 \times D_{10}$
131	7	145	714780	$C_6 \times L_2(19)$
134	S_3	141, 144	833910	J_1
137	5	145	1000692	$D_{12} \times L_2(19)$
138	5	145	1000692	$2^2 \times J_1$
139	2^2	144	1250865	$3 \times J_1$
141	3	145	1667820	$S_3 \times J_1$
142	3	145	1667820	$D_{20} \times L_2(19)$
144	2	145	2501730	$D_{10} \times J_1$
145	1	–	5003460	$J_1 \times L_2(19)$

Table 3: Semisparse Subgroups Corresponding to Section Regular Polytopes - Part 2

No.	Group	Subgroups	Facets	Aut($\mathcal{P}$)
5	$19:3\times D_{38}$	10, 14, 46, 47	$D^{2280}H^{60}$	3^2
6	$19\times 11:10$	12, 24, 35, 51	$D^{2376}H^{36}$	C_9
7	$2^3\times 19:9$	15, 21, 60	$D^{3640}H^{35}$	$7:3$
8	$19\times A_5$	31, 36, 49, 68	$D^{4344}H^{90}$	C_{18}
9	$19\times A_5$	31, 36, 50, 69	$D^{4344}H^{90}$	C_9
14	$19\times D_{38}$	26, 84, 85	$D^{6840}H^{180}$	$3\times C_9$
15	$(2^2)\times(19:9)$	27, 32, 87	$D^{7300}H^{30}$	C_6
16	$(2^2):(19:9)$	32, 39, 88	$D^{7300}H^{30}$	C_6
17	$(2^2):(19:9)$	32, 39, 88	$D^{7300}H^{30}$	C_6
18	$L_2(11)$	68, 69, 76, 115	$D^{7296}H^{570}$	$L_2(19)$
19	$19\times D_{30}$	28, 36, 50, 92	$D^{8688}H^{180}$	C_{18}
20	$C_{95}\times S_3$	28, 35, 49, 93	$D^{8760}H^{36}$	C_{18}
21	$2^2\times(19:3)$	32, 43, 98	$D^{10920}H^{105}$	$3\times 7:3$
24	$19\times D_{22}$	34, 85, 99	$D^{11880}H^{180}$	C_{45}
25	$19\times D_{20}$	35, 36, 37, 59, 104	$D^{13068}H^{198}$	C_9
27	$2\times 19:3$	38, 47, 110	$D^{14620}H^{20}$	A_5
29	$5\times(2^3:7)$	75, 83, 89	$D^{17784}H^{171}$	$2^2\times 3$
31	$19\times A_4$	59, 74, 117	$D^{21900}H^{90}$	C_{18}
32	$2^2\times(19:3)$	47, 59, 118	$D^{21900}H^{90}$	$3^2\times 2$
35	C_{190}	56, 85, 120	$D^{26316}H^{36}$	C_{18}
36	$19\times D_{10}$	56, 85, 123	$D^{26244}H^{180}$	C_{18}
37	$19\times D_{10}$	56, 85, 125	$D^{26244}H^{180}$	$S_3\times C_9$
41	$8:(7:3)$	75, 97, 101	$D^{29640}H^{285}$	$L_2(19)$
42	$(2^3:7)\times 3$	75, 98, 102	$D^{29640}H^{285}$	$3\times S_3$
43	19×2^3	59, 130	$D^{32760}H^{315}$	$9\times 7:3$
44	$2\times A_5$	68, 97, 104, 115	$D^{40812}H^{1767}$	$L_2(19)$
45	$19:6$	73, 84, 132	$D^{43320}H^{1140}$	$L_2(19)$
46	$3\times D_{38}$	72, 84, 133	$D^{43320}H^{1140}$	$3\times S_3$
47	$2\times 19:3$	71, 85, 133	$D^{43860}H^{60}$	$3\times A_5$
49	$19\times S_3$	74, 85, 135	$D^{43800}H^{180}$	$9\times D_{10}$
50	$19\times S_3$	74, 85, 136	$D^{43800}H^{180}$	C_{18}
51	$11:10$	76, 99, 120	$D^{45144}H^{684}$	$L_2(19)$
54	$5\times D_{22}$	77, 99, 124	$D^{45144}H^{684}$	$2^2\times 5$
59	$2^2\times 19$	85, 140	$D^{65700}H^{270}$	$C_6\times C_9$
60	$2^3\times 9$	87, 98	$D^{69160}H^{665}$	$2\times 7:3$
63	$5\times D_{14}$	89, 113, 124	$D^{71136}H^{684}$	$2^2\times 3$
65	$S_3\times D_{10}$	92, 93, 94, 104, 115	$D^{82080}H^{2622}$	$L_2(19)$
68	A_5	117, 123, 135	$D^{82536}H^{1710}$	$2\times L_2(19)$
69	A_5	117, 123, 136	$D^{82536}H^{1710}$	$L_2(19)$
70	$3\times D_{20}$	90, 91, 104, 118	$D^{82764}H^{1254}$	S_3

Table 4: The Remaining Semisparse Subgroups - Part 1

No.	Group	Subgroups	Facets	Aut($\mathcal{P}$)
75	$(2^3):7$	130, 131	$D^{88920}H^{855}$	$3\times L_2(19)$
80	$7:6$	101, 113, 132	$D^{118560}H^{1140}$	$L_2(19)$
81	$3\times D_{14}$	102, 113, 133	$D^{118560}H^{1140}$	$3\times S_3$
83	$2^3\times 5$	105, 130	$D^{124488}H^{1197}$	$2^2\times 7:3$
84	D_{38}	106, 143	$D^{129960}H^{3420}$	$3\times L_2(19)$
85	C_{38}	107, 143	$D^{131580}H^{180}$	$9\times A_5$
87	$2^2\times 9$	110, 118	$D^{138700}H^{570}$	$2^2\times 3$
88	$2^2:9$	118, 129	$D^{138700}H^{570}$	C_6
90	C_{30}	111, 120, 133	$D^{166668}H^{228}$	D_{12}
91	$3\times D_{10}$	111, 123, 133	$D^{166212}H^{1140}$	D_{12}
92	D_{30}	112, 123, 136	$D^{165072}H^{3420}$	$2\times L_2(19)$
93	$5\times S_3$	112, 120, 135	$D^{166440}H^{684}$	$2\times L_2(19)$
94	$3\times D_{10}$	112, 125, 132	$D^{166212}H^{1140}$	$2\times L_2(19)$
95	$5\times S_3$	124, 135	$D^{166440}H^{684}$	$2^2\times D_{10}$
97	$2\times A_4$	117, 130, 132	$D^{207480}H^{1995}$	$L_2(19)$
98	$2^3\times 3$	118, 130	$D^{207480}H^{1995}$	$S_3\times 7:3$
99	D_{22}	119, 143	$D^{225720}H^{3420}$	$5\times L_2(19)$
104	D_{20}	120, 123, 125, 140	$D^{248292}H^{3762}$	$L_2(19)$
105	$2^2\times 5$	124, 140	$D^{249660}H^{1026}$	$2^3\times 3$
110	C_{18}	128, 133	$D^{277780}H^{380}$	$2\times A_5$
113	D_{14}	131, 143	$D^{355680}H^{3420}$	$3\times L_2(19)$
115	D_{12}	132, 135, 136, 140	$D^{414960}H^{3990}$	$L_2(19)$
117	A_4	140, 142	$D^{416100}H^{1710}$	$2\times L_2(19)$
118	$2^2\times 3$	133, 140	$D^{416100}H^{1710}$	$C_6\times S_3$
120	C_{10}	137, 143	$D^{500004}H^{684}$	$2\times L_2(19)$
123	D_{10}	137, 143	$D^{498636}H^{3420}$	$2\times L_2(19)$
124	C_{10}	138, 143	$D^{500004}H^{684}$	$2^2\times A_5$
125	D_{10}	137, 143	$D^{498636}H^{3420}$	$S_3\times L_2(19)$
130	2^3	140	$D^{622440}H^{5985}$	$(7:3)\times L_2(19)$
132	C_6	142, 143	$D^{833340}H^{1140}$	$2\times L_2(19)$
133	C_6	141, 143	$D^{833340}H^{1140}$	$S_3\times A_5$
135	S_3	142, 143	$D^{832200}H^{3420}$	$D_{10}\times L_2(19)$
136	S_3	142, 143	$D^{832200}H^{3420}$	$2\times L_2(19)$
140	2^2	143	$D^{1248300}H^{5130}$	$C_6\times L_2(19)$
143	2	145	$D^{2500020}H^{3420}$	$A_5\times L_2(19)$

Table 5: The Remaining Semisparse Subgroups - Part 2